An Alternative Approach for Further Approximate Optimum Inspection Intervals

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Received Date, October 2006; Accepted Date, November 2007

Abstract. Having previously presented an article entitled “Further approximate optimum inspection intervals” in this Journal, here the author derives an alternative set of general explicit formulae using Cardan’s solution to a cubic equation and presents a modified heuristic algorithm for solving Baker’s model. The examples show that this new alternative approximate solution procedure for determining near optimum inspection intervals is as accurate and computationally efficient as the one suggested in the previous article. Through the examples, the author also indicates the relative merits and demerits of the two algorithms.

Keywords: Exponential Distribution, Inspection, Replacement, Cost, Profit, Machine.

1. INTRODUCTION

Consider a single unit representing a manufacturing system composed of many components. In the following, the author will use the word “machine” to refer to such a single-unit or complex system. Now suppose that a machine follows the exponential failure distribution \( F(t) = 1 - e^{-\lambda t} \) for a constant hazard \( \lambda > 0 \) and time \( t \geq 0 \), and that failures can be revealed only by periodic inspection (or testing) and then replaced. Frequent inspection increases inspection costs while infrequent inspection leads to increasing lost production costs. Thus, an economically optimum inspection interval usually exists. Recent studies in which the basic profit model, proposed by Baker (1990), of periodically inspecting a machine has been extended, generalized or modified can be found in Leung (2005). Recent articles concerning inspection problems are Yang and Klutke (2001), Lam (1995, 2003), Cui et al. (2004) and Zequeria and Berenguer (2006).

For easy reference, the author restates the essential equations of Baker’s model in the next section. In the rest of this article, he: (1) proposes an alternative near optimum solution procedure for Baker’s model; (2) gives three typical examples to show that this new alternative procedure is as accurate and computationally efficient as the one put forward by Leung (2005); (3) gives the relative merits and demerits of the two algorithms; (4) concludes with a possible application of the procedure; and (5) shows the limited applicability of a second or third degree Taylor series approximation for the factor \( e^{-z} \) arising in the maximum condition.

2. THE EXPECTED AND MAXIMUM PROFIT RATES, AND THE MAXIMUM CONDITION

Let \( a \) be the profit per unit time while the machine is operating and \( b \) be the cost of replacement if the machine is found to have failed, where \( a, b \geq 0 \). We assume that all replacements are equally expensive, that a failure completely halts production until the next inspection and replacement, and that each replacement restores the machine to the as-good-as new state. Let \( c \) be the cost of each periodic inspection, where \( c \geq 0 \). Now, suppose that the machine is inspected with periodic time \( T \) between two successive inspections. The expected profit rate (or per unit time) is given by Baker (1990):

\[
z(T) = \frac{P(T)}{T} = \frac{1}{T} \left[ \frac{a}{\lambda} - b \left( 1 - e^{-\lambda T} \right) - c \right].
\]  

(1)

The maximum condition of equation (1) is


\[(1 + x_a)e^{-\alpha x_a} = 1 - d, \quad (2.1)\]

or equivalently the condition can be written as the logarithmic form

\[\ln(1 + x_a) - x_a = \ln(1 - d), \quad (2.2)\]

where \(x_a = \lambda T_a\) and \(d = \frac{c}{x - b}\).

With \(\frac{c}{x} - b - c > 0\), Hariga (1996) showed the existence and uniqueness of the optimum inspection interval \(T_a\). However, the author thinks that his deduction is not direct and complete for the situation under study. Thus, the author states Theorem 1 below and gives its proof in the Appendix. Through economic interpretations of expressions \(a = \frac{c}{x} - b - c\) and \(\phi = \frac{c}{x} - b\), the new proof also provides more insight into the model.

**Theorem 1.** If \(\alpha > 0\) or equivalently \(a > \lambda(b + c)\), then there exists a unique finite optimum interval \(T_a\) that maximizes the expected profit rate \(z(T)\). Otherwise, \(T_a = \infty\), i.e. no inspections take place.

The maximum profit rate is

\[z(T_a) = \frac{a - (b + c)\lambda}{1 + \lambda T_a}. \quad (3)\]

The emphasis in this article is to find accurate approximate solutions \(x_a = \lambda T_a\) of equation (2.2) and then determine the maximum profit rate using equation (3).

### 3. More Accurate Approximate Optimum Inspection Intervals

A third degree Taylor series approximation for \(\ln(1+x)\) is given by

\[\ln(1 + x) \approx x - \frac{x^2}{2} + \frac{x^3}{3}. \quad (4)\]

Vaurio (1994) used the accurate approximation

\[\ln(1 + x) \approx \frac{x(1 + \frac{x}{2})}{1 + \frac{x}{2}}, \quad (5.1)\]

Equation (5.1) can be written as

\[\ln(1 + x) \approx x - \frac{x^2}{2} + \frac{x^3}{3 + 2x}. \quad (5.2)\]

Putting equation (5.2) into equation (2.2) yields a quadratic equation with the solution

\[x_a = \frac{-2}{3} \ln(1 - d) + \sqrt{\frac{4}{9} \ln^2(1 - d) - 2 \ln(1 - d)}. \quad (5.3)\]

Note that equation (5.3) corresponds to equation (8) in Vaurio (1994).

The author deduces from equations (4) and (5.2) that the general form of approximation for \(\ln(1+x)\) is given by

\[\ln(1 + x) \approx x - \frac{x^2}{2} + \frac{x^3}{3 + lx}, \quad \text{for} \quad 2 \leq l \leq 3. \quad (6.1)\]

Equation (6.1) can be written as

\[\ln(1 + x) \approx \frac{3x + (l - x^2)x^2 - (\frac{c}{x} - 1)x^3}{3 + lx}, \quad (6.2)\]

In particular, putting equation (6.2) with \(l = 2\) into equation (2.2), we can obtain equation (5.3).

In general, putting equation (6.2) with \(l > 2\) into equation (2.2) yields a cubic equation

\[(l - x_0^2)x_0^3 + 3x_0^2 + 2\ln(1 - d)lx_0 + 6\ln(1 - d) = 0. \quad (7)\]

The solution to equation (7) is given as follows:

Equating the coefficients of terms with the same order in equations (7) and (A3), we obtain

\[a_0 = l - 2, \quad a_1 = 1, \quad a_2 = \frac{c}{x} \ln(1 - d)l \quad \text{and} \quad a_3 = 6\ln(1 - d).\]

From equation (A5), we have

\[H = \frac{(l - x_0^2)\ln(1 - d)l - 1}{(l - x_0^2)^2} \quad \text{and} \quad G = \frac{6\ln(1 - d)l - 1(l - x_0^2)^2}{(l - x_0^2)^2}. \quad (8)\]

Since \(G^2 + 4H^3 < 0\) (the proof is given in the Appendix), from equation (A7) we have

\[\theta = \frac{1}{3} \cos^{-1}\left[\frac{-G}{2\sqrt{H}}\right], \quad (9)\]

and from equations (A4) and (A6) we have

\[x_i = 2\sqrt{-H} \cos \theta - \frac{1 - 2}{i - 2}. \quad (10)\]

In the Appendix, a brief discussion is given of the general solution of the cubic equation. For more details, see pp.131-133 in Tranter (1976).