MODULAR TRIBONACCI NUMBERS BY MATRIX METHOD

EunMi Choi

Abstract. In this work we study the tribonacci numbers. We find a tribonacci triangle which is an analog of Pascal triangle. We also investigate an efficient method to compute any n-th tribonacci numbers by matrix method, and find periods of the sequence by taking modular tribonacci number.

1. Introduction

The study of Fibonacci sequence $F_n$ ($n \geq 0$) has a long history since Lucas, 1885. The research has been extended to algebraic aspects, such as Fibonacci group([9], [4]) and Fibonacci ring[2], etc. It is also generalized to higher-order sequences including tribonacci[5], quatranacci, k-step Fibonacci sequences[1]. The 3-step Fibonacci sequence usually called the tribonacci sequence $T_n$ is the sum of the preceding three terms having initial values 0, 0, 1. Hence $T_n = T_{n-1} + T_{n-2} + T_{n-3}$ with $T_{-1} = T_0 = 0$ and $T_1 = 1$, so the first some numbers are $\{T_n\} : 0, 0, 1, 1, 2, 4, 7, 13, 24, 44, \cdots$.

The purpose of this work is to study the tribonacci numbers. We construct a tribonacci triangle which is an analog of Pascal triangle so that every tribonacci number appears in the triangle. We find an efficient method to compute any n-th tribonacci numbers by matrix method, and investigate periods of the sequence by taking modular tribonacci number.

2. Tribonacci Numbers with Binomial Coefficients

For the Fibonacci sequence $F_n$, it is known that if $M = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ then $M^n = \begin{bmatrix} F_2 & F_1 \\ F_1 & F_0 \end{bmatrix}^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$ thus $F_n^2 - F_{n-1}F_{n+1} = (-1)^{n-1}$. Fibonacci sequence

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is to \( M \) what tribonacci sequence is to \( N = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \), in fact

\[
\begin{bmatrix} T_{n+1} \\ T_n \\ T_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} T_n \\ T_{n-1} \\ T_{n-2} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^{n-1} \begin{bmatrix} T_2 \\ T_1 \\ T_0 \end{bmatrix}.
\]

**Theorem 2.1.** Let \( N \) be the matrix as above.

1. \( N = \begin{bmatrix} T_2 & 1 & T_1 \\ T_1 & 0 & T_0 \\ T_0 & 1 & T_{-1} \end{bmatrix} \) and \( N^n = \begin{bmatrix} T_{n+1} & T_n + T_{n-1} & T_n \\ T_n & T_{n-1} + T_{n-2} & T_{n-1} \\ T_{n-1} & T_{n-2} + T_n & T_{n-2} \end{bmatrix} \).

2. \( 1 = T_0^2 + T_1^2 - T_1 T_{-1} - T_2 T_0 \)

3. \( T_{n-1}^3 - 1 = 2T_{n-2}T_{n-1}T_n + T_{n-3}T_{n-1}T_n + T_{n-2}^2T_n + T_{n-3}T_n^2 - T_{n-2}T_n + T_{n-1}T_{n+1} \)

\[= T_{n-2}(2T_{n-1}T_n - T_{n+1}) + T_n(2T_n^2 - T_{n-1}T_{n+1}).\]

**Proof.** Since \( N^2 = \begin{bmatrix} T_3 & T_2 + T_1 & T_2 \\ T_2 & T_1 + T_0 & T_1 \\ T_1 & T_0 + T_{-1} & T_0 \end{bmatrix} \), (1) follows by induction. Moreover since

\[
1 = \det(N) = T_0^2 + T_1^2 - T_1 T_{-1} - T_2 T_0
\]

\[
= \det(N^n) = \begin{vmatrix} T_{n+1} & T_n + T_{n-1} & T_n \\ T_n & T_{n-1} + T_{n-2} & T_{n-1} \\ T_{n-1} & T_{n-2} + T_{n-3} & T_{n-2} \end{vmatrix} = \begin{vmatrix} T_{n+1} & T_n & T_{n-1} \\ T_n & T_{n-1} & T_{n-2} \\ T_{n-1} & T_{n-2} & T_{n-3} \end{vmatrix},
\]

we have

\[
T_{n+1}T_{n-2} - T_{n+1}T_{n-1}T_{n-3} + T_n^2T_{n-3} - 2T_nT_{n-2}T_{n-1} + T_{n-1}^3 = 1,
\]

hence \( T_{n-1}^3 - 1 = T_{n-3}(T_n^2 - T_{n-1}T_{n+1}) + T_n(2T_{n-1}T_n - T_{n+1}). \)

**Theorem 2.2.** \( T_{-n} = \begin{bmatrix} T_{n-1} \\ T_{n-2} \\ T_{n-1} \end{bmatrix} \) so \( T_{-n} \equiv T_{n-1}^2 \equiv (T_{n-2} + T_{n-3})^2 \) (mod \( T_n \)).

**Proof.** Since \( N^{-n} = (N^n)^{-1} \), it follows that

\[
\begin{bmatrix} T_{n+1} & T_n + T_{n-1} & T_n \\ T_n & T_{n-1} + T_{n-2} & T_{n-1} \\ T_{n-1} & T_{n-2} + T_{n-3} & T_{n-2} \end{bmatrix}^{-1} = \begin{bmatrix} T_{n-2} & T_{n-1} \\ T_{n-3} & T_{n-2} \\ T_{n-1} & T_n + T_{n-2} \end{bmatrix}^{-1} \begin{bmatrix} T_{n-2} & T_{n-1} \\ T_{n-3} & T_{n-2} \\ T_{n-1} & T_n + T_{n-2} \end{bmatrix}^{-1}.
\]