Translation numbers in Garside groups

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We show that all Garside groups are strongly translation discrete, that is, the translation numbers of non-torsion elements are strictly positive and for any real number \( r \) the number of conjugacy classes whose translation number is less than \( r \) is finite.

1 Introduction

Garside groups. In [Gar69], Garside solved the word and conjugacy problem in the braid groups using the lattice structure of the positive braid monoids. His theory has been generalized and improved by several mathematicians [BS72, Thu92, EM94, BKL98, DP99, BKL01, FG03, Pic01b, Geb03]. Garside group, introduced by Dehornoy and Paris in [DP99], is a lattice theoretic generalization of the braid groups and the Artin groups of finite type. Therefore most of the results in the braid groups using Garside theory can be naturally extended to Garside groups. For example, Garside groups have normal form, have solvable conjugacy problem and are biautomatic.

Translation numbers. For a group \( \mathcal{G} \) and a finite generating set \( X \), the translation number of \( g \in \mathcal{G} \) is defined by

\[
t_X(g) = \lim_{n \to \infty} \frac{|g^n|_X}{n},
\]

where \( |.|_X \) denotes the minimal word length in the alphabet \( X^{\pm 1} \). A group \( \mathcal{G} \) is said to be translation proper if the translation numbers of non-torsion elements are strictly positive; translation discrete if the translation numbers of non-torsion elements are bounded away from 0; strongly translation discrete if for any real number \( r \), there are only finitely many conjugacy classes of \( g \in \mathcal{G} \) with \( t_X(g) \leq r \). In [GS91], Gersten and Short first introduced the notion of translation number, and showed that biautomatic groups are translation proper. Several interesting results follow from it: Polycyclic subgroups (hence finitely generated nilpotent subgroups) of a biautomatic group are abelian by finite; A Baumslag-Solitar group

\[
B_{k,l} = \langle x, y \mid yx^ky^{-1} = x^l \rangle
\]

cannot be isomorphic to a subgroup of a biautomatic group if \( |k| \neq |l| \).

The word hyperbolic groups have stronger discreteness properties of the translation numbers [Gro87]. The translation numbers are rational with bounded denominators and, moreover, every non-torsion element has a positive power which is conjugate to a straight element. \( g \in \mathcal{G} \) is said to be straight if \( |g^n|_X = n |g|_X \) for all \( n \in \mathbb{Z} \).

In [Bes99], Bestvina showed that Artin groups of finite type are translation proper. In [CMW04], Charney, Meier and Whittlesey generalized Bestvina's result to Garside groups, but they put an extra condition of tameness.

The results. In this paper, we show that all Garside groups are strongly translation discrete. Our approach is different from that of Bestvina. We remark that we don’t need the tameness of Garside groups and our result is stronger than that of Bestvina even for Artin groups of finite type.

2 Garside groups

Here we collect relevant information about the word and conjugacy problem in Garside groups. See [DP99, Deh02, FG03, Pic01b] for details.

Let \( M \) be a monoid. Let atoms (or 'indivisible elements') be the elements of \( M \) such that \( a \neq 1 \) and if \( a = bc \) then either \( b = 1 \) or \( c = 1 \). Let \( ||a|| \) be the supremum of the lengths of all expressions of \( a \) in terms of atoms. The monoid \( M \) is said to be atomic if it is generated by its atoms and \( ||a|| < \infty \) for any \( a \in M \).

In an atomic monoid \( M \), there are partial orders \( \leq_L \) and \( \leq_R \): \( a \leq_L b \) if \( ac = b \) for some \( c \in M \); \( a \leq_R b \) if \( ca = b \) for some \( c \in M \). For \( a \in M \), let

\[
L(a) = \{ b \in M : b \leq_L a \} \quad \text{and} \quad R(a) = \{ b \in M : b \leq_R a \}.
\]

Definition 2.1. An atomic monoid \( M \) is called a Garside monoid if it satisfies the following:

(i) \( M \) is finitely generated;
(ii) \( M \) is left and right cancellative;
(iii) \( (M, \leq_L) \) and \( (M, \leq_R) \) are lattices;
(iv) There exists an element \( \Delta \), called a Garside element, such that \( L(\Delta) \) and \( R(\Delta) \) are the same and they form a set of generators for \( M \).

The set \( L(\Delta) = R(\Delta) \) is usually denoted \( \mathcal{D} \). The elements of \( \mathcal{D} \) are called the simple elements. The gcd and
A Garside group is defined to be the group of fractions of a Garside monoid. Because Garside monoids satisfy the Ore condition, they embed in their group of fractions. When $M$ is a Garside monoid and $G$ the group of fractions of $M$, we identify the elements of $M$ and their images in $G$ and call them the positive elements of $G$. $M$ is called the positive monoid of $G$, often denoted $G^\ast$. The partial orders $\leq_L$ and $\leq_R$, and thus the lattice structures in the positive monoid $G^\ast$ can be extended to the Garside group $G$ as follows: $g \leq_L h$ (respectively, $g \leq_R h$) for $g, h \in G$ if $gh = h(g)$ (respectively, $ag = h$) for some $a \in G^\ast$. For $g \in G$, the invariants $\inf(g)$, $\sup(g)$ and $\lcm(g)$ are defined as follows:

$$\inf(g) = \max\{r \in \mathbb{Z} : \Delta^r \leq_L g\},$$
$$\sup(g) = \min\{s \in \mathbb{Z} : g \leq_L \Delta^s\},$$
$$\lcm(g) = \sup(g) - \inf(g).$$

Let $\tau : G \to G$ be the inner automorphism of $G$ such that $\tau(g) = \Delta^{-1} g \Delta$.

**Theorem 2.2.** Let $G$ be a Garside group, $G^\ast$ its positive monoid and $\Delta$ a Garside element.

(i) $\tau : G \to G$ induces an automorphism of $G^\ast$.

(ii) There is an integer $k$ such that $\Delta^k$ is central in $G$.

(iii) For any $g \in G$, there are integers $r$ and $s$ such that $\Delta^r \leq_L g \leq_L \Delta^s$.

(iv) For any $g \in G$ has a unique expression

$$g = \Delta^r s_1 \cdots s_k,$$

where $s_1, \ldots, s_k \in \mathcal{D} \setminus \{1, \Delta\}$ and $(s_i s_{i+1} \cdots s_k) \wedge_L \Delta = s_i$ for $i = 1, \ldots, k$. In this case, $\inf(g) = r$ and $\sup(g) = r + k$.

The expression in (iv) is called the normal form of $g$.

For $g \in G$, let $[g]$ denote its conjugacy class. Let

$$\inf_{\ast}(g) = \max\{\inf(h) : h \in [g]\},$$
$$\sup_{\ast}(g) = \min\{\sup(h) : h \in [g]\},$$
$$\lcm_{\ast}(g) = \sup_{\ast}(g) - \inf_{\ast}(g).$$

The super summit set $[g]^S$ of $g$ is defined by

$$[g]^S = \{h \in [g] : \inf(h) = \inf_{\ast}(g), \sup(h) = \sup_{\ast}(g)\}.$$  

For $g \in G$ with normal form $g = \Delta^r s_1 \cdots s_k$, the cyclings $c(g)$ and the decyclings $d(g)$ are defined by

$$c(g) = \Delta^r s_2 \cdots s_k \tau^{-1}(s_1),$$
$$d(g) = \Delta^r \tau^k(s_k) s_1 \cdots s_{k-1}.$$  

**Theorem 2.3.** Let $G$ be a Garside group and $g \in G$.

(i) $[g]^S$ is finite and non-empty.

(ii) An element of $[g]^S$ can be obtained by applying a finite sequence of cyclings and decyclings to $g$.

(iii) If $h \in [g]^S$, then $c(h), d(h), \tau(h) \in [g]^S$.

(iv) If $h \in [g]^S$, then $\tau(c(h)) = c(\tau(h))$ and $\tau(d(h)) = d(\tau(h))$.

(v) For any $h, h' \in [g]^S$, there is a finite sequence

$$h = h_0 \to h_1 \to \cdots \to h_m = h'$$

such that for $i = 1, \ldots, m$, $h_i \in [g]^S$ and $h_i = s_i h_{i-1} s_i^{-1}$ for some $s_i \in \mathcal{D}$.

In [Geb03], Gebhardt introduced the ultra summit set

$$[g]^U = \{h \in [g]^S : c^k(h) = h \text{ for some } k > 0\}$$

and showed that Theorem 2.3 is true when we replace the super summit set $[g]^S$ with the ultra summit set $[g]^U$.

### 3 The wreath product of a Garside group

Let $G$ and $H$ be Garside groups and $G$ (and hence $G^\ast$) act on $H^\ast$ (from right) via a homomorphism $\rho : G \to \text{Aut}(H^\ast)$, where $\text{Aut}(H^\ast)$ denotes the group of automorphisms of the monoid $H^\ast$. We use the notation $b^g$ to denote $\rho(g)(b)$, the action of $g \in G$ on $b \in H^\ast$ via $\rho$. Recall that the semidirect product $G \ltimes \rho H$ has the underlying set

$$\{(a, b) : a \in G^\ast, b \in H^\ast\}$$

and the product

$$(a_1, b_1)(a_2, b_2) = (a_1 a_2, b_1^g b_2)$$

for $a_1, a_2 \in G^\ast$ and $b_1, b_2 \in H^\ast$. Composing $\rho$ with the canonical monomorphism $\text{Aut}(H^\ast) \to \text{Aut}(H)$, we get a homomorphism $\rho' : G \to \text{Aut}(H)$, hence the semidirect product $G \ltimes \rho' H$ is also well-defined. If there is no confusion, we omit $\rho$ and $\rho'$ in the notation of semidirects.

We need the fact that the semidirect product of two Garside monoids is a Garside monoid. In fact, it is already known by Picantin [Pic01a, Proposition 3.12]. He showed the crossed product of Garside monoids is a Garside monoid. Our semidirect product is a special case of his crossed product. Therefore we state the following theorem without proof.

**Theorem 3.1.** If $G$ and $H$ are Garside groups and $\rho : G \to \text{Aut}(H^\ast)$ is a homomorphism, then $G^\ast \ltimes \rho H^\ast$ is a Garside monoid with the group of fractions $G \ltimes \rho H$. Moreover, if $\Delta_G$ and $\Delta_H$ are Garside elements of $G$ and $H$ and $\Delta_H$ is fixed under the action of $G$, then $(\Delta_G, \Delta_H)$ is a Garside element of $G^\ast \ltimes \rho H^\ast$ and the set of simple elements is $\{(a, b) : a \leq_L \Delta_G, b \leq_L \Delta_H\}$.

**Definition 3.2.** For a Garside group $G$, the wreath product $G \wr \mathcal{G}$ is the semidirect product $G^\ast \ltimes \mathcal{G}$, where $\mathcal{G} = \langle \delta \rangle$ acts on the cartesian product $G^n$ by

$$(g_1, \ldots, g_n) \delta = (g_n, g_1, \ldots, g_{n-1}).$$