A Study on the Bi-Aspect Test for the Two-Sample Problem

Seung-Man Hong\textsuperscript{a}, Hyo-Il Park\textsuperscript{1,b}

\textsuperscript{a}Department of Informational Statistics, Korea University
\textsuperscript{b}Department of Statistics, Chongju University

Abstract
In this paper we review a bi-aspect nonparametric test for the two-sample problem under the location translation model and propose a new one to accommodate a more broad class of underlying distributions. Then we compare the performance of our proposed test with other existing ones by obtaining empirical powers through a simulation study. Then we discuss some interesting features related to the bi-aspect test with a comment on a possible expansion for the proposed test as concluding remarks.

Keywords: Bi-aspect test, combining function, nonparametric test, permutation principle.

1. Introduction
When one considers a comparison study for a treatment with control, one may try to apply a suitable nonparametric test when the underlying distribution cannot be assumed with any specific one. All the nonparametric test statistics are directly related with ranks that may alleviate some severe departure from the usual observations that may be called as outliers. Because of minimal assumptions for the underlying distributions, it may well be that the powers of resulting nonparametric tests are lower than those of parametric ones under some specific distributions. Then in order to enhance the power of test, one may consider to use several test procedures simultaneously (\textit{cf.} Park, 2011a, 2011b). This procedure may be called the versatile test. In addition, one may try to reduce the scope of the null hypothesis by splitting the null hypothesis into sub-hypotheses and then intersecting the splitted sub-hypotheses. Pesarin (2001) initiated this procedure and named it as the multi-aspect test. This test procedure has been developed and expanded in various situations by many authors. Marozzi (2004a) considered a bi-aspect test procedure for the location parameter for the two-sample problem and expanded it for the multi-sample case (\textit{cf.} Marozzi, 2004b). In addition, for testing equality of two distributions in a case-control design with treatment effects, Salmaso and Solari (2005) considered several different features of a null hypothesis that leads to the multiple-aspect test.

In order to complete the chosen test, one has to derive the null distribution of the test statistics to obtain the critical value for any given significance level or \(p\)-value for a more general conclusion of the test. However, the derivation of the null distribution may be difficult if not impossible since the used statistics may be correlated to each other in a complicated manner. One way out this quagmire would be to use a re-sampling method such as the bootstrap or permutation method. This approach may depend heavily on the computer ability and applicable software.

In this research, we propose a new bi-aspect nonparametric test for the two-sample problem under the location translation model. In the next section we review Marozzi’s result and propose a new counterpart for broad applications to various distributions. We consider the use of permutation principle.
for obtaining $p$-values. Then we compare the performance of the proposed test with Marozzi’s that includes some individual tests (through a simulation study) and discuss interesting features related to the bi-aspect tests as concluding remarks.

2. Bi-Aspect Test

Let $X_{11}, \ldots, X_{1n_1}$ and $X_{21}, \ldots, X_{2n_2}$ be two independent random samples from populations with continuous but unknown distribution functions $F_1$ and $F_2$, respectively. Then we consider the following location translation model such that for some $\delta \in (-\infty, \infty)$,

$$F_2(x) = F_1(x + \delta), \quad \text{for all } x \in (-\infty, \infty).$$

Under this model, Marozzi (2004a, 2004b) proposed bi-aspect nonparametric test procedures for testing the null hypothesis

$$H_0 : \{\delta = 0\} \cap \{F_1 = F_2\}$$

based on the following two statistics $T_1$ and $T_2$ such that

$$T_1 = \sum_{i=1}^{n_1} X_{1i} \quad \text{and} \quad T_2 = \sum_{i=1}^{n_2} I(X_{1i} > \bar{M}),$$

where $I(\cdot)$ is an indicator function and $\bar{M}$ is a sample median from the combined sample. We note that $T_1$ is a version of two-sample $t$-statistic and $T_2$, the Mood-type median test statistic (cf. Mood, 1950). This means that for testing $H_{01} : \delta = 0$ and $H_{02} : F_1 = F_2$, one may use $T_1$ and $T_2$ as test statistics, respectively. For the time being, for more detailed discussion of our arguments, we now consider the one-sided alternative such that $H_{11} : \delta > 0$ or $H_{12} : F_1(x) < F_2(x)$ for some $x \in (-\infty, \infty)$. Let $\lambda_1$ and $\lambda_2$ be the respective p-values for testing $H_{01} : \delta = 0$ against $H_{11} : \delta > 0$ and $H_{02} : F_1 = F_2$ against $H_{12} : F_1(x) < F_2(x)$ for some $x \in (-\infty, \infty)$ with $T_1$ and $T_2$. Then by choosing a suitable combining function to obtain an overall p-value, one can continue this testing procedure as follows. With the Tippett combining function (cf. Pesarin, 2001), Marozzi (2004a, 2004b) proposed a test statistic $T_{12}$ based on $T_1$ and $T_2$ as follows.

$$T_{12} = \max \{1 - \lambda_1, 1 - \lambda_2\}.$$  

For each $j$, $j = 1, 2$, we note that the p-value $\lambda_j$ is a random variable since $\lambda_j$ is a function from $T_j$. Since at least any one of $\lambda_j$’s tends to be 0 if $H_0 : \{\delta = 0\} \cap \{F_1 = F_2\}$ is false, the test based on $T_{12}$ would be to reject $H_0$ when the value of $T_{12}$ approaches 1. Then in order to complete this test procedure, we need the null distribution of $T_{12}$. For this, Marozzi (2004a, 2004b) applied the permutation principle (cf. Good, 2000; Pesarin, 2001) for obtaining the overall p-value for $T_{12}$. Especially, Marozzi (2004a) presented a procedure by obtained the p-value with the Monte-Carlo approach and modified slightly the form of statistics by adding 1/2 and 1 in the numerator and denominator, respectively to ensure to obtain the p-values in the interval because of computational convenience. Since the statistic $T_1$ is a version of t-statistic that produces an optimal test when the underlying distribution is normal, one may worry about the power of the test when the underlying distribution may be skewed or heavy-tailed. Thus the test based on $T_{12}$ may be inappropriate when the underlying distribution is exponential or Cauchy. For this reason, we may propose a new test that uses the Wilcoxon rank-sum