Stationary Bootstrap Prediction Intervals for GARCH($p, q$)

Eunju Hwang, Dong Wan Shin

Institute of Mathematical Sciences and Department of Statistics, Ewha Womans University

Abstract
The stationary bootstrap of Politis and Romano (1994) is adopted to develop prediction intervals of returns and volatilities in a generalized autoregressive heteroskedastic (GARCH($p, q$)) model. The stationary bootstrap method is applied to generate bootstrap observations of squared returns and residuals, through an ARMA representation of the GARCH model. The stationary bootstrap estimators of unknown parameters are defined and used to calculate the stationary bootstrap samples of volatilities. Estimates of future values of returns and volatilities in the GARCH process and the bootstrap prediction intervals are constructed based on the stationary bootstrap; in addition, asymptotic validities are also shown.

Keywords: GARCH model, stationary bootstrap, prediction, asymptotics.

1. Introduction
For accessing the risks of financial assets, valid predictions of future returns as well as future volatilities are essential. For that purpose, among many time series models, the generalized autoregressive heteroskedastic (GARCH) model is one of the most successful one. Baillie and Bollerslev (1992) studied prediction error distribution in the GARCH(1, 1) model with known parameter values. Andersen and Bollerslev (1998) provided an empirical study to show that the GARCH model is good for predicting future volatilities. Andersen et al. (2001) proposed models and methods to predict realized volatilities. Engle and Patton (2001) compared several volatility models including GARCH(1, 1) for forecasting volatilities.

In order to have more understanding of the risks, we need to know sampling distributions of predictions from which we can construct prediction intervals, Value at Risk (VaR), and others. Even though GARCH models allow the simple constructions of predictions, the mathematical analysis of the sampling distribution would be difficult if the estimated GARCH parameters are plugged-in in constructing the predictions. All the above studies for the prediction assumed given parameter values.

Bootstrapping methods are practical alternatives to mathematical analysis in understanding sampling distributions of predictions that address variations due to estimated parameters. Miguel and Olave (1999) and Reeves (2005) proposed bootstrapping ARCH($p$) prediction intervals for future returns. Pascual et al. (2006) developed GARCH(1, 1) prediction intervals for returns and volatilities. Chen et al. (2011) developed a computationally efficient GARCH(1, 1) bootstrap prediction intervals for returns and volatilities.

However, all the above studies of the bootstrap GARCH predictions did not provide mathematical justification for their bootstrapping methods. Our aim is to construct a mathematically valid bootstrapping method for prediction intervals of future returns and future volatilities. The validity is more
important for risk analysis because an invalid method would provide misleading information about risk. For example an invalid method for VaR would not guarantee the given level (1% or 5%, for example).

We adopt the stationary bootstrapping proposed by Politis and Romano (1994). The stationary bootstrapping is a block resampling method with an increasing random block length that has attracted several authors with recent applications, such as Swensen (2003), Paparoditis and Politis (2005), Parker et al. (2006) for unit root tests, and Hwang and Shin (2011, 2012a) for nonparametric analyses. Lahiri (1999), Nordman (2009) and Hwang and Shin (2012b) analyzed new properties of the stationary bootstrap. In our stationary bootstrapping method, the dependence structure of the original data is transferred to the bootstrapping sample by resampling blocks. On the other hand, in the bootstrappings of Miguel and Olave (1999), Reeves (2005), Pascual et al. (2006), Chen et al. (2011) for GARCH predictions, the dependence structure is transferred to the bootstrapping sample by iterating the GARCH model using normally bootstrapped residuals.

Our method utilizes squares of GARCH processes that have ARMA representations which in turn have long-AR representations. Under the ARMA+AR representation, parameter estimation is linear, which allows us to develop a mathematically valid bootstrapping procedure.

We consider a stationary GARCH\((p, q)\) process defined by (2.1) and (2.2) below. The GARCH model is represented as a form of ARMA. Given a realization of the past observations, the ordinary least square estimators of the ARMA model via long-AR approach are first constructed and residuals are computed. The stationary bootstrap method is applied to generate stationary bootstrap sample of returns, and these data are used to develop the stationary bootstrap estimators, from which the bootstrap sample of volatilities are calculated; in addition, the classical bootstrap residuals are used for estimated \(i.i.d\). errors. It is shown that the stationary bootstrap estimators have the same limiting distribution of the ordinary least squares estimators.

Using the stationary bootstrap estimators and the bootstrap samples of returns and volatilities, we obtain bootstrap future values of returns and volatilities of the GARCH process. Slutsky’s Theorem and mathematical induction shows that the sampling distribution of the stationary bootstrap future values converges in probability to the distribution of the unknown future value. The stationary bootstrap prediction intervals for returns and volatilities of the GARCH process are constructed by means of bootstrap replicates and quantiles of the Monte Carlo estimates of the bootstrap distributions. By the fundamental weak consistency, the asymptotic validities of the stationary bootstrap prediction intervals are established.

The paper is organized as follows. In Section 2, the stationary bootstrap procedure is described. An algorithm is given for developing the stationary bootstrap estimator of unknown parameters and for constructing the stationary bootstrap prediction intervals of future values of the GARCH process. In Section 3, asymptotic results of the estimators and the prediction intervals are given along with the proofs.

2. Stationary Bootstrapping and Prediction Intervals

We consider a GARCH\((p, q)\) process, \(\{y_t\}_{t=1}^T\), (for \(p, q \geq 1\)), satisfying, for \(t = 1, \ldots, T\)

\[
y_t = \sigma_t \epsilon_t,
\]

\[
\sigma_t^2 = \alpha_0 + \sum_{i=1}^{p} \alpha_i y_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2.
\]