An Empirical Study on Explosive Volatility Test with Possibly Nonstationary GARCH(1, 1) Models

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Abstract

In this paper, we implement an empirical study to test whether the time series of daily returns in stock and Won/USD exchange markets is strictly stationary or explosive. The results indicate that only a few series show nonstationary volatility when dramatic events erupted; in addition, this nonstationary behavior occurs more often in the Won/USD exchange market than in the stock market.

Keywords: GARCH model, Lyapunov exponent, strict stationarity testing.

1. Introduction

Time varying volatility is an important feature of financial time series. To capture this phenomenon, Engle (1982) introduced the idea of conditional heteroscedasticity. Since then, generalized autoregressive conditional heteroscedasticity (GARCH) models have drawn significant attention from researchers. The stationarity is a basic assumption for GARCH models; however, the stationarity test has not been intensively studied yet: see Francq and Zakoian (2012). Some articles consider weakly stationary GARCH models (Bollerslev, 1986; Weiss, 1986; Pantula, 1988); however, in practice, the parameter estimation result often appears to violate the weak stationarity condition since the underlying process turns out to be integrated GARCH (IGARCH). It is well known that the IGARCH process is not weakly stationary but strictly stationary: see Nelson (1990). To cover the IGARCH case, Lee and Hansen (1994) and Lumsdaine (1996) derived asymptotic results for GARCH(1, 1) models without the weak stationarity assumption; in addition, Bougerol and Picard (1992) verified a necessary and sufficient condition for GARCH(p, q) process to be strictly stationary by applying the theory of products of random matrices and the top Lyapunov exponent. It is noteworthy that the region of parameters (to allow the strict stationarity) is larger than that for the weak stationarity. Berkes et al. (2003), Francq and Zakoian (2004), and Straumann and Mikosch (2006) established the asymptotic properties of quasi-maximum likelihood estimators (QMLE) for GARCH-type models under the strict stationarity assumption: see also Li et al. (2002).

According to Nelson (1990), the conditional variance of GARCH(1, 1) process explodes to the infinity if the process is not strictly stationary and the intercept is positive. Recently, Jensen and Rahbek (2004) and Francq and Zakoian (2012) studied the asymptotic properties of Gaussian-QMLE for nonstationary GARCH(1, 1) processes. Francq and Zakoian (2012) also proposed a strict stationarity
test based on the estimator of the Lyapunov exponent and showed that their test has a robust feature against the model misspecification.

In this paper, we analyze 11 series of daily asset returns ranging from 1996 to 2012. We perform a strict stationarity test in each year to investigate how often the time series show nonstationary behavior such as explosive volatility. In Section 2, we review the conditions for the strict stationarity of GARCH processes. Further, the asymptotic theory for the nonstationary GARCH models and the strict stationarity test are summarized. In Section 3 provides, an empirical study is provided.

2. Strict Stationarity of GARCH Processes

This section reviews the strict stationarity of GARCH processes and introduces a recent method for the stationarity test.

2.1. Lyapunov exponent

The GARCH(1, 1) model is defined by

\[ \varepsilon_t = \sqrt{h_t}, \quad h_t = \omega + \alpha \varepsilon_{t-1}^2 + \beta h_{t-1}, \]

where \( \omega > 0, \alpha \geq 0, \beta \geq 0, \) and \( \{\eta_t\} \) is a sequence of i.i.d. random variables with \( E\eta_t = 0 \) and \( E\eta_t^2 = 1. \) The Equation (2.1) can be interpreted as the one defining a first order homogeneous Markov process \( \{(\varepsilon_t, h_t) : t = 0, 1, 2, \ldots\}, \) of which the state space is \( \mathbb{R} \times (0, \infty) \) and the initial state is \((\varepsilon_0, h_0).\)

Bollerslev (1986) verified that the Equation (2.1) admits a weakly stationary solution if and only if \( \alpha + \beta < 1. \) Theorem 2 of Nelson (1990) shows that the Equation (2.1) has a unique strictly stationary solution if and only if

\[ E \left[ \log \left( \beta + \alpha \eta_t^2 \right) \right] < 0. \]  

(2.2)

It is noteworthy that Theorem 2 of Nelson (1990) does not require the restrictions \( E\eta_t = 0 \) and \( E\eta_t^2 = 1. \) It is only assumed that \( E[\log(\beta + \alpha \eta_t^2)] \) exists, for which \( E \log^+ \eta_t^2 < \infty \) is sufficient where \( \log^+ x = \max(\log x, 0): \) see Lemma 2.2 of Straumann and Mikosch (2006).

Let us recall the linkage of (2.1) and (2.2). Model (2.1) involves the following stochastic recurrence equation (SRE): for \( t \geq 1, \)

\[ h_t = \omega + (\beta + \alpha \eta_{t-1}^2) h_{t-1}. \]  

(2.3)

Subsequent substitution yields that

\[ h_t = \omega \left\{ 1 + \sum_{k=1}^{t-1} (\beta + \alpha \eta_{k-1}^2) \cdots (\beta + \alpha \eta_2^2) \right\} + (\beta + \alpha \eta_{t-1}^2) \cdots (\beta + \alpha \eta_0^2) h_0. \]

By the law of large numbers, the condition (2.2) implies that there exists \( \delta > 0 \) such that \( \prod_{i=1}^{k} (\beta + \alpha \eta_{i-1}^2) = O(e^{-\delta k}) \) with probability one, so \( \sum_{k=1}^{\infty} \prod_{i=1}^{k} (\beta + \alpha \eta_{i-1}^2) \) converges a.s. Theorem 2 of Nelson (1990) verified that (2.2) is a necessary and sufficient condition for the convergence of the series. Then, we can see that \( h_{t, \infty} := \omega \left\{ 1 + \sum_{k=1}^{\infty} (\beta + \alpha \eta_{k-1}^2) \cdots (\beta + \alpha \eta_2^2) \right\} \) is well-defined and is a strictly stationary solution to the SRE (2.3). Assuming \( h_0 = h_{0, \infty}, \) the bivariate Markov process \( \{(\varepsilon_t, h_t) : t = 0, 1, 2, \ldots\} \) is strictly stationary, so its time domain can be extended to \( \mathbb{Z} \) by Kolmogorov\'s extension theorem; see Billingsley (1995). Further, the stationary solution to the Equation (2.1) is explicitly expressed as \( \varepsilon_t = h_{t, \infty}^{-1/2} \) and \( \{(\varepsilon_t, h_t) : t \in \mathbb{Z}\} \) is ergodic by Theorem 36.4 of Billingsley (1995).